The 8 th Week Lectures

1. Pull-back of a function

Let $\Phi : G \to D$ be a C^1-map from region G to D in $\mathbb{R}^2$.

$$\Phi(u,v) = (x,y), \quad x = g(u,v), \quad y = h(u,v), \text{ where}$$

g, h are C^1, ie, $\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial u}, \frac{\partial h}{\partial v}$ are continuous in G.

For $f = f(x,y)$ in D, its pull-back via Φ is a fcn defined in G given by

$$\hat{f}(u,v) = f(g(u,v), h(u,v)).$$

For instance, $f(x,y) = x^2 y + \log(x+y)$, $x = u-v$, $y = u+v$.

$$\hat{f}(u,v) = (u-v)^2(u+v) + \log(2u).$$

2. Jacobian Matrix and the Jacobian

The jacobian matrix of Φ =

$$J_\Phi = \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix}$$

The Jacobian (determinant) is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det J_\Phi = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}.$$

e.g. $\Phi = (g, h)$, $g(u,v) = u+v$, $h(u,v) = u-v$.

then
$$J\Phi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

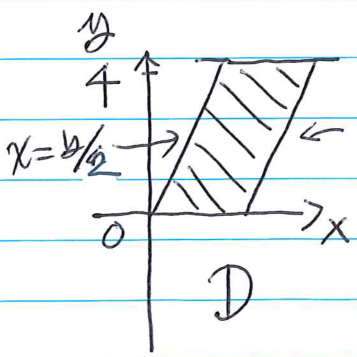
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1\times(-1) - 1\times 1 = -2.$$

3. Change of Variables Formula.

Theorem Let $\Phi : G \to D$ be C^1-map which maps G onto D and is 1–1 on its interior. Then

$$\iint_D f(x,y)\, dA(x,y) = \iint_G \hat{f}(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v).$$

e.g. 1. Evaluate $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\, dx\, dy$.



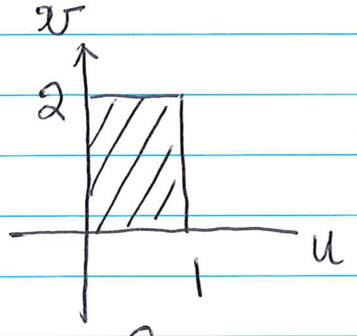
$\leftarrow x = y/2 + 1$ First, realize that

$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\, dx\, dy = \iint_D \frac{2x-y}{2}\, dA(x,y)$$

$$D = \{(x,y) : 0 \le y \le 4, \; y/2 \le x \le y/2 + 1\}$$

Let $u = x - \frac{y}{2}$, $v = \frac{y}{2}$. Solving for x, y in terms of u, v:

$x = u + v$, $y = 2v$.



Then $\Phi(u,v) = (u+v, 2v)$ maps G 1-1 onto D. Here $G = [0,1] \times [0,2]$.

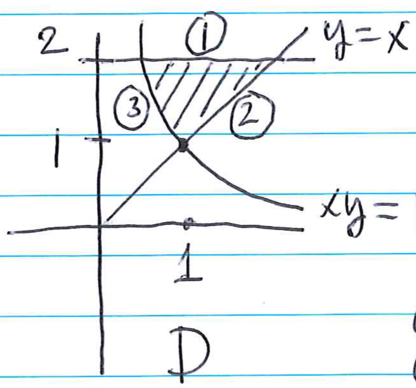
$$\therefore \iint_D \frac{2x-y}{2}\, dA(x,y) = \iint_G u \, \frac{\partial(x,y)}{\partial(u,v)}\, dA(u,v).$$

As $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 > 0$,

$$\iint_G u \times 2 \times dA(u,v) = \int_0^2 \int_0^1 2u \, du\, dv$$
$$= 2.$$

$$\therefore \int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\, dx\, dy = 2 \quad \#$$

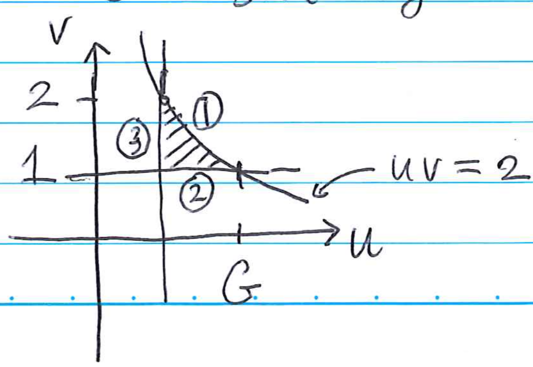
e.g. 2 $\displaystyle\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$.



Let $u = \sqrt{xy}$, $v = \sqrt{\frac{y}{x}}$ ie

$x = u/v$, $y = uv$.

G is given by



① $\mapsto$ ①
② $\mapsto$ ②
③ $\mapsto$ ③
boundary corresp.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v} > 0$$

$$\therefore \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint_D \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dA(x,y)$$

$$= \iint_G v e^{u} \frac{2u}{v}\, dA(u,v)$$

$$= \iint_G 2u e^{u}\, dA(u,v)$$

$$= \int_1^2 \int_1^{\frac{2}{u}} 2u e^{u}\, dv\, du$$

$$\vdots$$

$$= 2e(e-2) \ \#$$

"Steps in Using the Formula"

(I) From the iterated integral to recover D.

(II) Introduce (u,v) and express (x,y) in terms of (u,v).

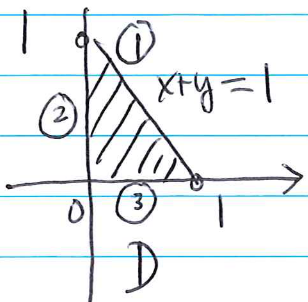
(III) Calculate $\frac{\partial(x,y)}{\partial(u,v)}$.

(IV) Determine G and plug in the formula

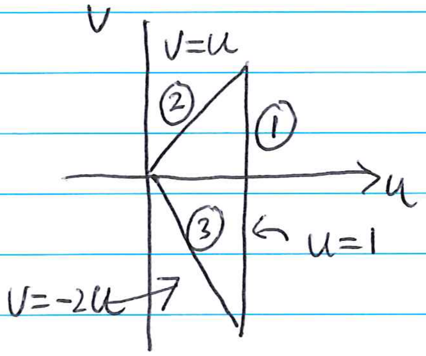
$$\iint_D f(x,y)\, dA(x,y) = \iint_G f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v).$$

e.g. 3 $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

(I)



D



G

let

$u = x+y,\ v = y-2x.$

that's

$x = \frac{u}{3} - \frac{v}{3},\quad y = \frac{2u}{3} + \frac{v}{3}$ (II)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$$ (III)

$\therefore$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$ (IV)

$$= \iint_D \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

$$= \iint_G \sqrt{u}\,v^2\,\frac{1}{3}\,dA(u,v)$$

$$= \int_0^1 \int_{-2u}^{u} u^{\frac{1}{2}} v^2 \frac{1}{3}\,dv\,du$$

$$= \frac{2}{9} \quad \#$$

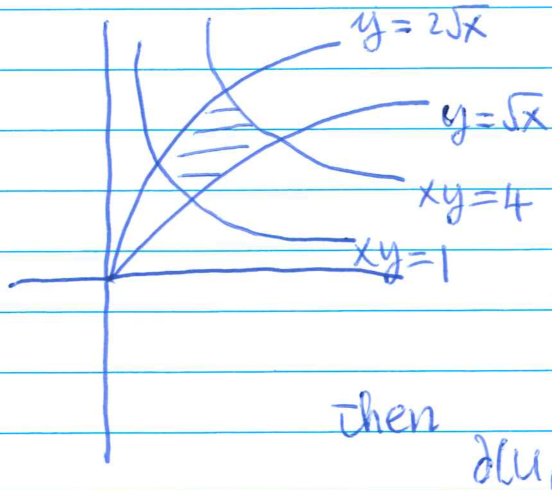
A trick:

step (II) may be simplified by using the relation

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}.$$

e.g. 4. Find the area of the region bounded by

$xy = 1, 4$; $y = \sqrt{x}, 2\sqrt{x}$.



$$I = \iint_D 1 \, dA(x,y)$$

(II) Let $u = xy \in [1,4]$

$v = \sqrt{x}$

then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ \frac{1}{2}x^{-\frac{1}{2}} & 0 \end{vmatrix} = -\frac{1}{2}\sqrt{x} = \frac{-v}{2}.$$

(III) $\therefore \dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{2}{v}$.

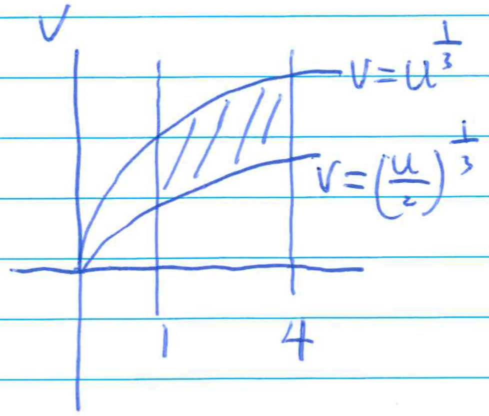
(IV) G: $u \in [1,4]$. $y = \sqrt{x}$ turns to $\frac{u}{x} = \sqrt{x}$, $u = v^3$,

$y = 2\sqrt{x}$ turns to $\frac{u}{x} = 2\sqrt{x}$, $u = 2v^3$.

G is described by

$$1 \le u \le 4$$

$$\left(\frac{u}{2}\right)^{1/3} \le v \le u^{\frac{1}{3}}$$



$$\therefore I = \iint_G \left|\frac{-2}{v}\right| dA(u,v)$$

$$= \int_1^4 \int_{\left(\frac{u}{2}\right)^{1/3}}^{u^{\frac{1}{3}}} \frac{2}{v} \, dv \, du$$

$$= \int_1^4 2\left[\log u^{\frac{1}{3}} - \log\left(\frac{u}{2}\right)^{\frac{1}{3}}\right] du$$

$$= 2\log 2 \ \#$$

In $n=3$, (x,y,z) now fcns of (u,v,w), the Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

e.g. 5 Evaluate $\displaystyle\int_0^3 \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\,dy\,dz.$

Let $u = x - \frac{y}{2}$, $v = \frac{y}{2}$, $w = \frac{z}{3}$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{6}.$$

$\therefore \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 6.$

$\therefore \Omega = 0 \le z \le 3,\ 0 \le y \le 4,\ \frac{y}{2} \le x \le \frac{y}{2}+1$

turns to : $0 \le w \le 1,\ 0 \le v \le 2,\ 0 \le u \le 1$

$\therefore$ The iterated integral is equal to

$$\int_0^1 \int_0^2 \int_0^1 (u+w)\, 6\, du\, dv\, dw$$

$= \cdots = 12$ #

Finally, we verify that the spherical coordinates

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho \cos\varphi.$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \cos\varphi \begin{vmatrix} \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$-(-\rho\sin\varphi) \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$= \cos\varphi \; \rho^2 (\cos\varphi \sin\varphi \cos^2\theta + \cos\varphi\sin\varphi\sin^2\theta)$$
$$+ \rho\sin\varphi\,(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi \sin^2\theta)$$
$$= \rho^2\cos^2\varphi\sin\varphi + \rho^2\sin\varphi\sin^2\varphi$$
$$= \rho^2\sin\varphi.$$

We conclude that the formula in spherical coordinates is a special case of the change of variables formula.